2ND ORDER CIRCUITS

RLC
First

Will get D.E. of Form

$$C_1 \frac{d^2F(t)}{dt^2} + C_2 \frac{dF(t)}{dt} + C_3 F(t) = g(t)$$

C_1 C_2 C_3 constants $g(t)$ known.
To solve completely for $F(t)$
will need to know $F(t=0)$
and $\left.\frac{dF}{dt}\right|_{t=0}$.

Useful Fact $V(0^+) = V(0^-)$ capacitor

$\cancel{\Rightarrow \left.\frac{dV}{dt}\right|_{t=0} = 0}$

$i(0^+) = i(0^-)$ inductor

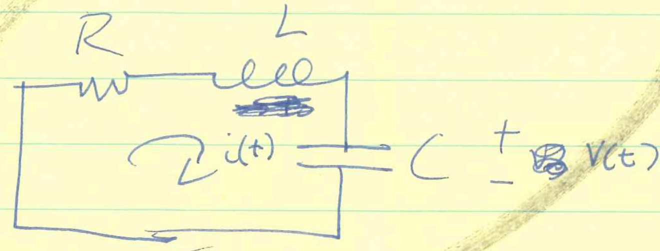
$\cancel{\Rightarrow \left.\frac{di}{dt}\right|_{t=0} = 0}$

$\omega = \sqrt{a}$

$\ddot{F} = aF$ soln $\cos(\omega t)$, $\sin(\omega t)$

$\Rightarrow F = \cos(\omega t)$

RLC



RINGING FIRST

$V(t=0) = V_0$

$i(t=0) = I_0$

RINGING A BELL
PLUCKING GUITAR STRING
ALL SAME MATH

KVL

$$i(t)R + L\frac{di(t)}{dt} + V(t) = 0$$

$$\Rightarrow \frac{di(t)}{dt}R + L\frac{d^2 i(t)}{dt^2} + \frac{dV(t)}{dt} = 0$$

$$q(t) = CV(t)$$

$$i(t) = \frac{dq(t)}{dt} = C\frac{dV(t)}{dt}$$

$$\Rightarrow \frac{dV(t)}{dt} = \frac{1}{C}\, i(t)$$

$$\boxed{\frac{di(t)}{dt}R + L\frac{d^2 i(t)}{dt^2} + \frac{1}{C}\, i(t) = 0}$$

$$\Rightarrow \boxed{\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0}$$

Need to know $i(t=0)$ & $\left.\frac{di}{dt}\right|_{t=0}$

$i(t=0) \equiv I_0$

what about $\left.\frac{di}{dt}\right|_{t=0}$?

$$KCL\Big|_{t=0} \quad \underbrace{i(t=0)}_{I_0} R + L \left.\frac{di(t)}{dt}\right|_{t=0} + \underbrace{V(t=0)}_{V_0} = 0$$

$$\Rightarrow \frac{di(t=0)}{dt} = -\frac{1}{L}\left(R I_0 + V_0\right)$$

How to solve 2ND order DE?

Again, if we find $i(t)$ that

1) Solves D.E.

2) Solves initial values,

that solution is unique.

So "how we get there" less important than result

~~ASA~~ Answer is:

$$i(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

A_1 A_2 constants, determined by initial conditions.

$$S_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad S_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

We will try guessing $i(t) = Ae^{st}$

A, s Constants.

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

$$\Rightarrow s^2 \cancel{A}\cancel{e^{st}} + \frac{R}{L} s \cancel{A}\cancel{e^{st}} + \frac{1}{LC} \cancel{A}\cancel{e^{st}} = 0$$

$$\Rightarrow s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \quad (*)$$

Guess ~~on~~ works for <u>any</u> A but only for ~~s~~ s soln. of $(*)$

$$\Rightarrow s_1 = \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = \frac{-R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Convenient to <u>define</u> $\omega_0 = \sqrt{\frac{1}{LC}}$ "Natural frequency" $\alpha = \frac{R}{2L}$ "Damping factor"

$$\Rightarrow s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$\Rightarrow i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

(4)

$\left(4\frac{1}{2}\right)$

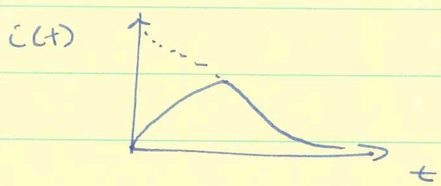
3 cases

"over damped" $\alpha > \omega_0$

$\Rightarrow s_1 \quad -ve$

$s_2 \quad -ve$

$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

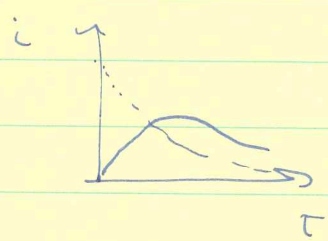


"Critically damped" $\alpha = \omega_0$

$\Rightarrow s_1 = s_2 \; \cancel{\Rightarrow i(t)} = \alpha$

Our guess was wrong.

$i(t) = \cancel{A_1 t} \, (A_2 + A_1 t) e^{-\alpha t}$



"Underdamped" $\alpha < \omega_0$

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$

$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

$j \equiv \sqrt{-1}$ $\quad \omega_d \equiv \sqrt{\omega_0^2 - \alpha^2}$ "damped nat. frequency"

$$\Rightarrow i(t) = A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t}$$

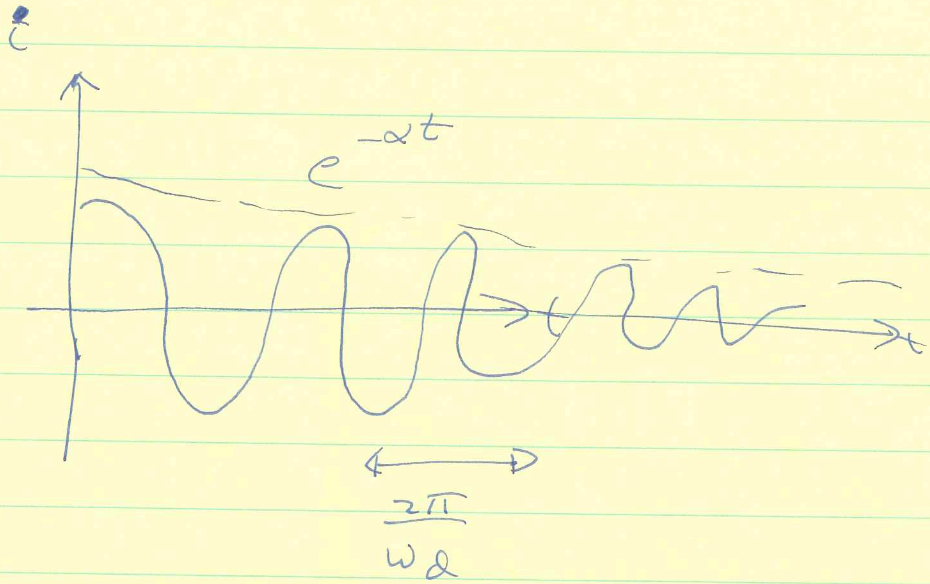
$$= e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad e^{-j\theta} = \cos\theta - j\sin\theta$$

$$i(t) = e^{-\alpha t}\left[A_1\left(\cos(\omega_d t) + j\sin(\omega_d t)\right) + A_2\left(\cos(\omega_d t) - j\sin(\omega_d t)\right)\right]$$

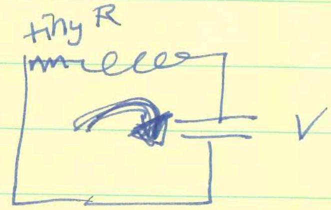
$$= e^{-\alpha t}\left[\underbrace{(A_1 + A_2)}_{B_1}\cos(\omega_d t) + \underbrace{j(A_1 - A_2)}_{B_2}\sin(\omega_d t)\right]$$

$$i(t) = e^{-\alpha t}\left[B_1\cos(\omega_d t) + B_2\sin(\omega_d t)\right]$$



1st before math

"Ringing" "Damping".



			Energy	
			$\frac{1}{2}Li^2$ ~~Ene~~	$\frac{1}{2}CV^2$
Initially	$i=0$	$V\neq 0$	0	$\neq 0$

$V\neq 0 \Rightarrow i\uparrow$

$i\neq 0 \Rightarrow V\downarrow$

$V\downarrow \Rightarrow \cancel{i\downarrow}$

$i\uparrow \Rightarrow q\downarrow$

$i\uparrow \Rightarrow \frac{di}{dt}\uparrow$

$\frac{di}{dt}\uparrow \Rightarrow V\downarrow$

$\frac{1}{2}CV^2\downarrow$

$\frac{1}{2}LI^2\uparrow$

Eventually	$V=0$	$I\neq 0$	$\neq 0$	0

~~$V = 0$ $i \neq 0$~~ $\frac{1}{2}LI^2$ $\frac{1}{2}CV^2$

$\neq 0$ $= 0$

$i \neq 0 \Rightarrow q \uparrow$

$q \uparrow \Rightarrow V \uparrow$

Energy goes back and forth between $\frac{1}{2}CV^2$ and $\frac{1}{2}LI^2$

Frequency is ω_0 or ω_D.

If $R=0$, no loss of energy.

If $R=0$, always loosing some energy, eventually get all energy gone $\Rightarrow$ $i=0$ $V=0$ $\Rightarrow$

Called "damping"

Damping rate R/L $\tau = L/R$

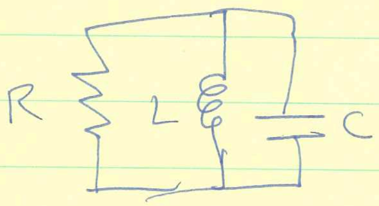
Freq $\omega_0 = \frac{1}{\sqrt{LC}}$

LATER

CAN DRIVE RESONANCE LIKE A SWING

END

|| RLC



Given I_0 thru L at $t=0$

V_0 across C at $t=0$

KCL

$$\frac{v(t)}{R} + i_L(t) + i_c(t) = 0$$

$$i_c(t) = C\frac{dv(t)}{dt}$$

$$\Rightarrow \quad \frac{v(t)}{R} + i_L(t) + C\frac{d\,v(t)}{dt} = 0$$

$$\frac{dv}{dt}\frac{1}{R} + \frac{di}{dt} + C\frac{d^2v}{dt^2} = 0$$

$$\frac{dv}{dt}\frac{1}{R} + \frac{v}{L} + C\frac{d^2v}{dt^2} = 0$$

$$\Rightarrow \quad \boxed{\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0}$$

$$V(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

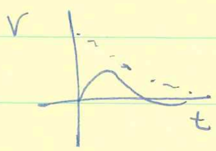
or $\quad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

Again 3 cases

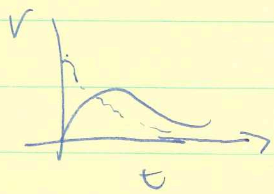
Overdamped $\quad \alpha > \omega_0$

$$V(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad s_1\ s_2\ \ -ve$$



Critical damp $\quad \alpha = \omega_0$

$$V(t) = (A_1 + A_2 t)\, e^{-\alpha t}$$



Underdamped $\quad \alpha < \omega_0$

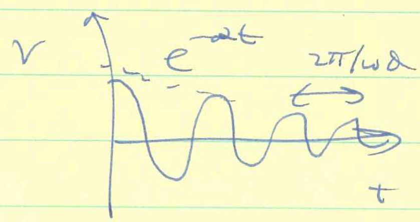
$$S_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$

$$S_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

$$j = \sqrt{-1} \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$S_{1,2} = -\alpha \pm j\omega_d \qquad \omega_d \equiv \sqrt{\omega_0^2 - \alpha^2}$$

$$V(t) = e^{-\alpha t} A_1 (\cos \omega_d t + B_1 \sin \omega_d t)$$



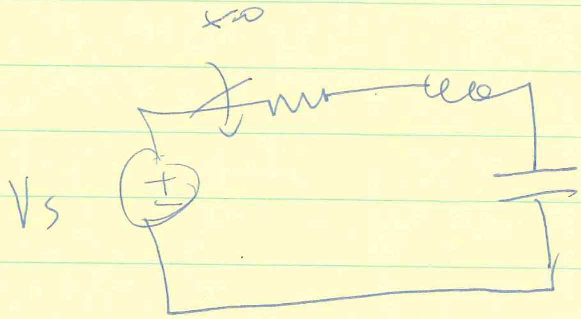
To Find A_1, A_2 need $V(0)$ and $\left.\frac{dV}{dt}\right|_{t=0}$

$V(0) = V_0$ known

$$(*)(*) \Rightarrow \quad \frac{V_0}{R} + I_0 + C \left.\frac{dV}{dt}\right|_{t=0} = 0$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{t=0} = \frac{-(V_0 + RI_0)}{RC}$$

RLC step response



KVL $L\frac{di}{dt} + Ri + v = V_s$ $\quad t>0$

$= 0 \quad t<0$

$q = Cv \Rightarrow i = C\,dv/dt$

$$\Rightarrow \frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC} \quad t>0$$

$$= 0 \quad t<0$$

$$\Rightarrow \frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = u(t)\,\frac{V_s}{LC}$$

later will by $\sin(\omega t)$

Soln. $v(t) = v_t(t) + v_{ss}(t)$
transient steady state

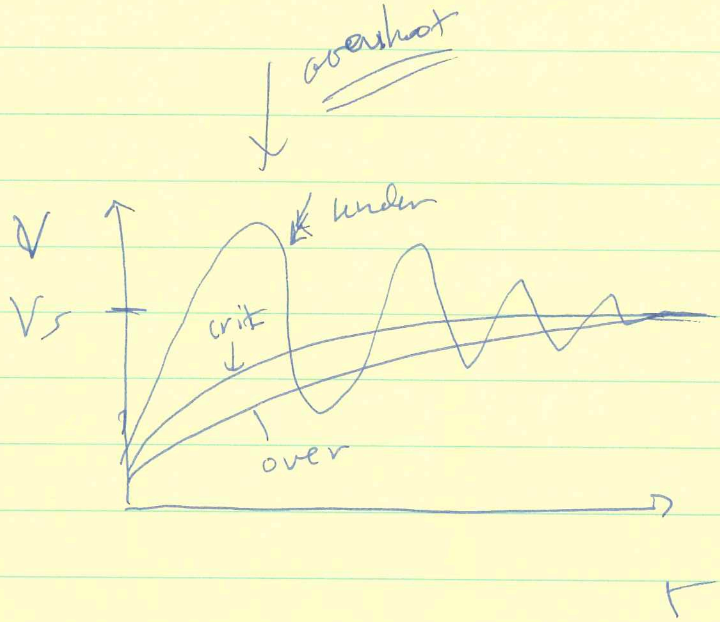
$v_t(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ ova

$(A_1 + A_2 t)\, e^{-\alpha t}$ crit

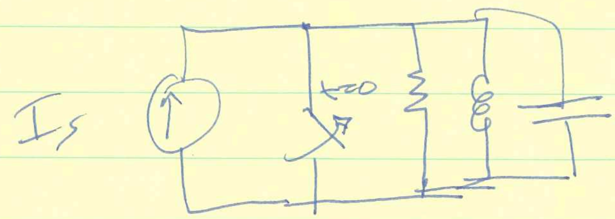
$(A_1 \cos\omega_d t + A_2 \sin\omega_d t)\, e^{-\alpha t}$ unde

$V_{ss}(t) = V(\infty) = V_s$

Prove.


overshoot
under
V
Vs
crit
over
t

|| RLC step



KCL $\frac{v}{R} + i + C\frac{dv}{dt} = I_s$

Using $v = L\,di/dt$

$\Rightarrow$

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$

$$i(t) = i_t(t) + i_{ss}(t)$$

$$i(t) = I_{ss} + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$ over

$$i(t) = I_{ss} + (A_1 + A_2 t)e^{-\alpha t}$$ crit

$$i(t) = I_{ss} + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t}$$ under

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TA will cover 2^{nd} order RC circuits
+ RL circuits

e.g. ..