Quantum mechanics of free electrons

- Important for quantized resistance calculation
- Important for single electron transistors
- Density of states
  - 3 dimensions
  - 2 dimensions
  - 1 dimensions
  - 0 dimensions
- Dimensionality (effective)
  - Set by size of nano-device compared to electron wavelength

Readings for this lecture

- Ferry, *Quantum Mechanics for Electrical Engineering*, ch. 1 (in handout packet)
- Hanson p. 16-44,62-69,85-101,chapter 8
- Good references:
  - Brandsen and Joachian, *Introduction to Quantum Mechanics*, Longman Scientific, 1989
  - Kittel, *Introduction to Solid State Physics*, Wiley, 1996
  - Ashcroft/Mermin, *Solid State Physics*, Saunders College, 1976

Quantum mechanics of free particles

$$\left|\Psi(\vec{r},t)\right|^2$$

is probability of finding an electron at point r at time t.

Ψ is complex, and both real and imaginary parts are physical.

Quantum mechanics of free particles:

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Ψ is complex, and both real and imaginary parts are physical.

$$\omega = E / \hbar$$

For a free particle:

$$\Psi(\vec{r},t) \sim e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Momentum:

$$\vec{p} = \hbar\vec{k}$$

Energy:

$$E = \frac{p^2}{2m} = \frac{(\hbar k)^2}{2m}$$

Schrodinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)$$

(1 dimension)

(Time dependent)

Let

$$\Psi(x,t) = A\cdot e^{i(kx-\omega t)}$$

A is a (complex) constant.

Then

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = i\hbar\frac{\partial}{\partial t}A\cdot e^{i(kx-\omega t)} = i\hbar(-i\omega)A\cdot e^{i(kx-\omega t)}$$

$$= E\cdot A\cdot e^{i(kx-\omega t)} = E\cdot\Psi(x,t)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(A\cdot e^{i(kx-\omega t)}\right) = \left(-\frac{\hbar^2}{2m}\right)(ik)^2\left(A\cdot e^{i(kx-\omega t)}\right)$$

$$= \frac{\hbar^2 k^2}{2m}\left(A\cdot e^{i(kx-\omega t)}\right) = \frac{p^2}{2m}\Psi(x,t)$$

Schrodinger equation:

(3 dimensions)

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = -\frac{\hbar^2}{2m}\vec{\nabla}^2\Psi(\vec{r},t) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Psi(\vec{r},t)$$

Let $\Psi(\vec{r},t) = A\cdot e^{i(\vec{k}\cdot\vec{r}-\omega t)} = A\cdot e^{i\left((k_x\cdot x+k_y\cdot y+k_z\cdot z)-\omega t\right)}$

Then $i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = i\hbar(-i\omega)\Psi(\vec{r},t) = E\cdot\Psi(\vec{r},t)$ as before.

But:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Psi(\vec{r},t) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(A\cdot e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right)$$

$$=\left(-\frac{\hbar^2}{2m}\right)\left((ik_x)^2+(ik_y)^2+(ik_z)^2\right)\left(A\cdot e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right) = \left(\frac{\hbar^2(k_x{}^2+k_y{}^2+k_z{}^2)}{2m}\right)\Psi(\vec{r},t)$$

$$=\frac{\hbar^2k^2}{2m}\left(A\cdot e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right) = \frac{p^2}{2m}\Psi(\vec{r},t)$$

Quantum mechanics of free particles:

$$\Psi(\vec{r},t) \sim e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Generally,

$$\Psi(\vec{r},t) = \sum_n A_n e^{i(k_n x-\omega_n t)} \rightarrow \int dk A(k) e^{i(kx-\omega t)}$$

is also a possibility.

Time-independent Schrodinger equation

$$\Psi(\vec{r},t) = A \cdot e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= A \cdot e^{i\left((k_x \cdot x + k_y \cdot y + k_z \cdot z) - \omega t\right)} = \underbrace{A \cdot e^{i(k_x \cdot x + k_y \cdot y + k_z \cdot z)}}_{\text{Call this } \psi(\vec{r})} \cdot e^{-i\omega t}$$

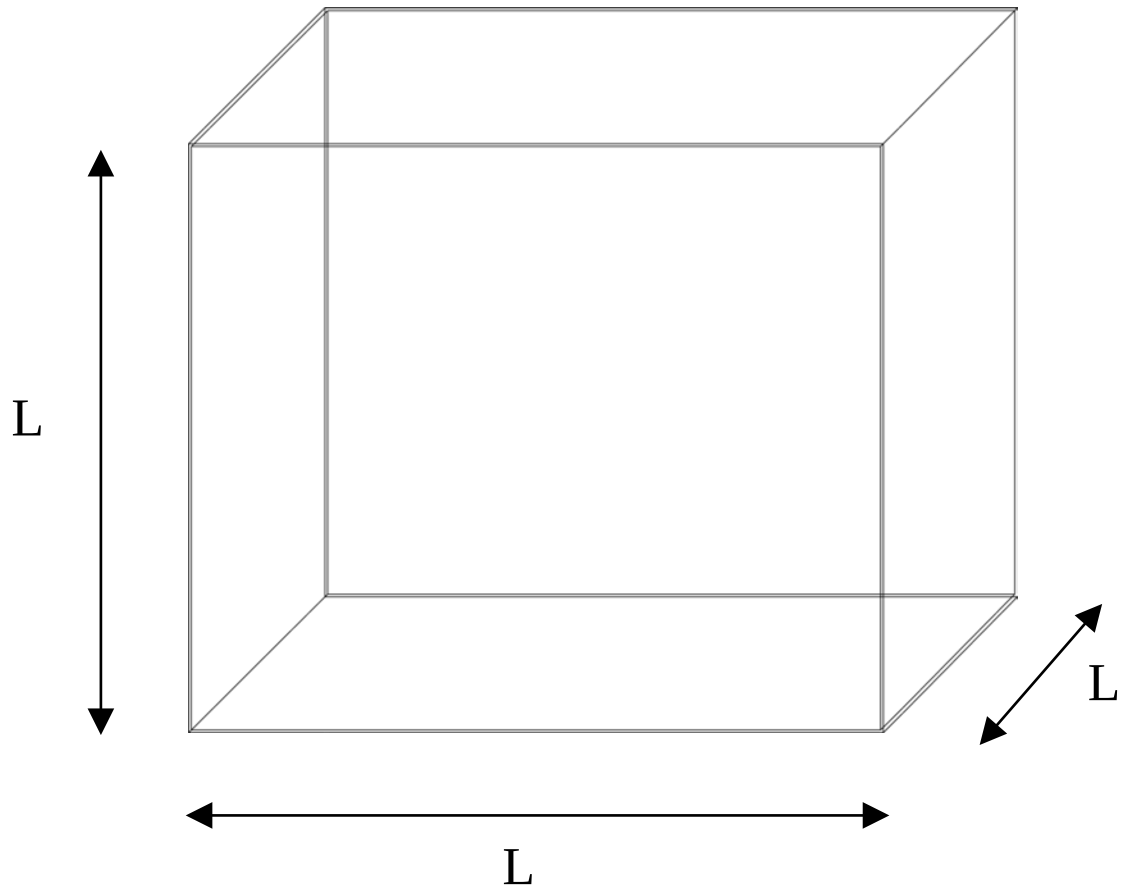
$$\Rightarrow \Psi(\vec{r},t) = \psi(\vec{r}) \cdot e^{-i\omega t}$$

From: $$i\hbar \frac{\partial}{\partial t}\Psi(\vec{r},t) = -\frac{\hbar^2}{2m}\vec{\nabla}^2\Psi(\vec{r},t)$$

$$i\hbar \frac{\partial}{\partial t}\Psi(\vec{r},t) = i\hbar \frac{\partial}{\partial t}\psi(\vec{r}) \cdot e^{-i\omega t} = i\hbar(-i\omega)\psi(\vec{r}) \cdot e^{-i\omega t} = E \cdot \psi(\vec{r}) \cdot e^{-i\omega t} = -\frac{\hbar^2}{2m}\vec{\nabla}^2\Psi(\vec{r},t) = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{r}) \cdot e^{-i\omega t}$$

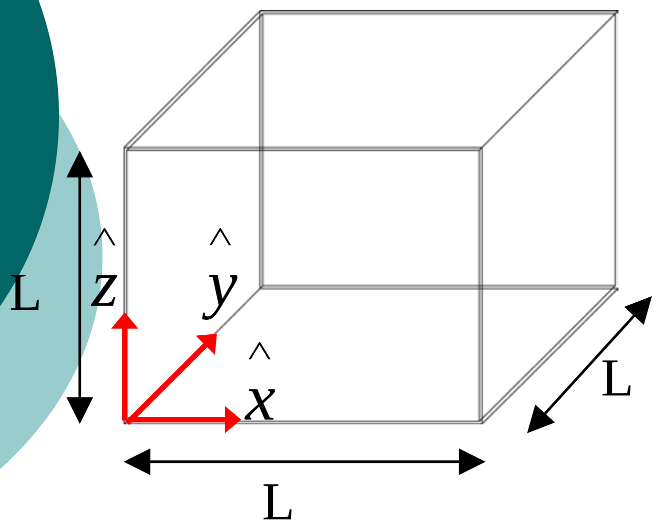
$$\boxed{\Rightarrow -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{r}) = E \cdot \psi(\vec{r})}$$

Confined particles: A box



Goal: find $\psi(\vec{r})$

Similar to electric field inside the box.



Goal: find $\psi(\vec{r})$

Everywhere outside the box

$$\left|\psi(\vec{r})\right|^2 = 0$$

In particular,

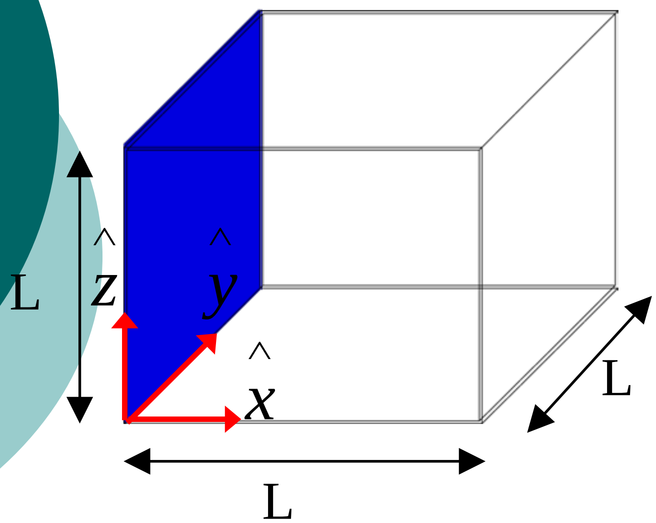
$$\left|\psi(\vec{r})\right|^2 = 0$$

on the boundaries.

As before, we will consider all six surfaces:

Boundary conditions:

The plane x=0:



Try:

$$\psi(\vec{r}) = A \cdot e^{i(k_x \cdot x + k_y \cdot y + k_z \cdot z)}$$

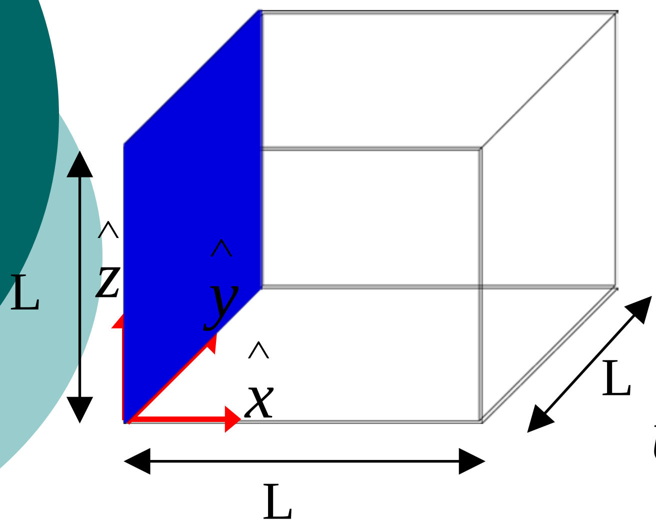
$$\psi(x=0, y, z) = A \cdot e^{i(k_x \cdot x + k_y \cdot y + k_z \cdot z)} = A \cdot e^{i(k_y \cdot y + k_z \cdot z)}$$

0

Does not solve boundary condition!!!

Boundary conditions:

The plane x=0:



Let's try something:

$$\psi(\vec{r}) = A \cdot e^{i(k_x \cdot x + k_y \cdot y + k_z \cdot z)}$$

$$-A \cdot e^{i(-k_x \cdot x + k_y \cdot y + k_z \cdot z)}$$

$$\psi(\vec{r}) = A \cdot \left(e^{ik_x \cdot x} - e^{-ik_x \cdot x} \right) \cdot e^{i(k_y \cdot y + k_z \cdot z)}$$

$$e^{a \cdot b} = e^a \cdot e^b$$

$$\psi(x=0, y, z) = A \cdot \left(e^{ik_x \cdot x} - e^{-ik_x \cdot x} \right) \cdot e^{i(k_y \cdot y + k_z \cdot z)}$$

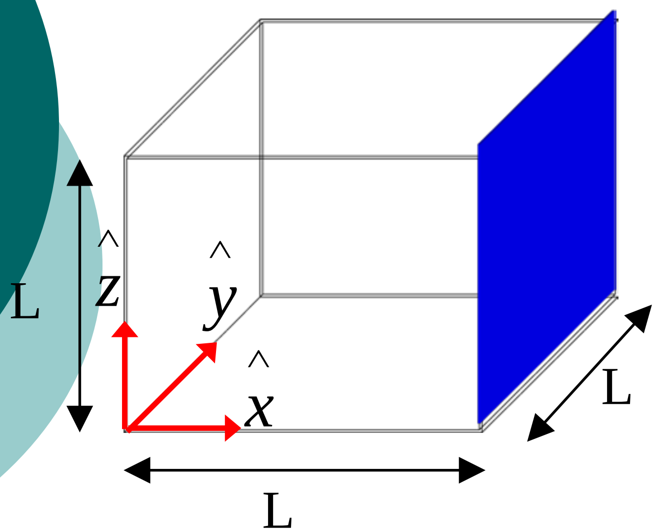
0 0

$$= A \cdot \left(e^0 - e^0 \right) \cdot e^{i(k_y \cdot y + k_z \cdot z)} = 0$$

Does solve boundary condition!!!

Boundary conditions:

The plane x=L:



$$\psi(\vec{r}) = A \cdot \left(e^{ik_x \cdot x} - e^{-ik_x \cdot x}\right) \cdot e^{i(k_y \cdot y + k_z \cdot z)}$$

$$= 2iA \cdot \sin(k_x x) \cdot e^{i(k_y \cdot y + k_z \cdot z)}$$

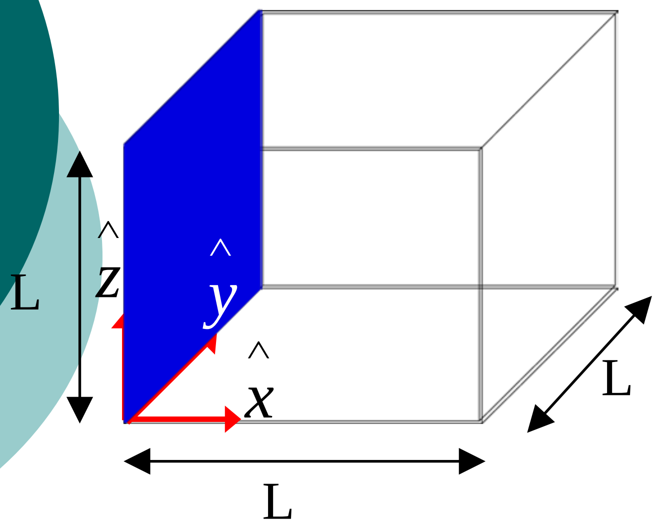
$$\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$\psi(x = L, y, z) = 2iA \cdot \sin(k_x L) \cdot e^{i(k_y \cdot y + k_z \cdot z)} = 0\,?$$

If and only if:

$$k_n = n\pi / L \quad n = 1, 2, 3 \ldots$$

Boundary conditions:



We can do the same for y, z:

$$\psi(\vec{r}) = (2i)^3 A \cdot \sin(k_{n_x} x) \cdot \sin(k_{n_y} y) \cdot \sin(k_{n_z} z)$$

$$k_{n_x} = n_x \pi / L$$

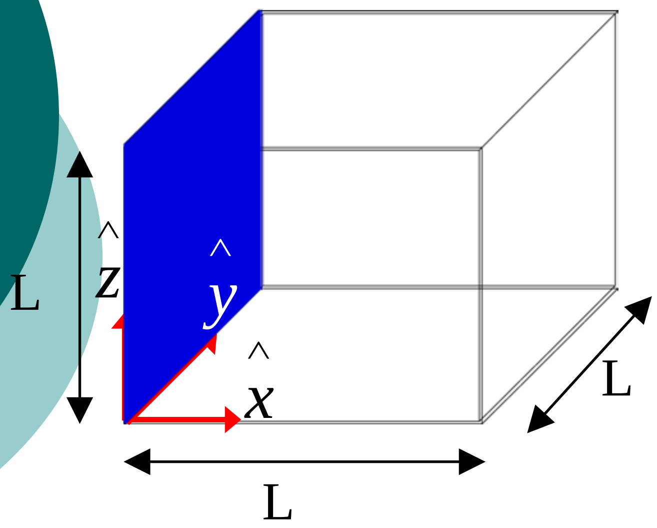
$$k_{n_y} = n_y \pi / L$$

$$k_{n_z} = n_z \pi / L$$

$$E = \frac{\hbar^2 (k_{n_x}{}^2 + k_{n_y}{}^2 + k_{n_z}{}^2)}{2m} = \frac{\hbar^2 (\pi / L)^2}{2m} (n_x{}^2 + n_y{}^2 + n_z{}^2)$$

These are the allowed energy levels, or “quantum states”

Many electrons:

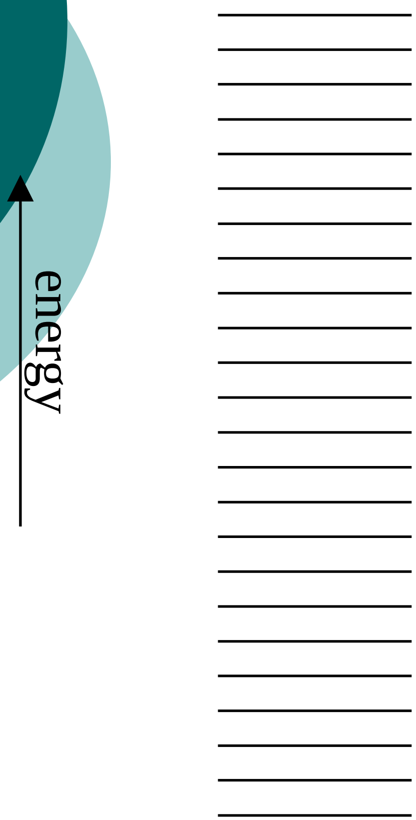


$$E = \frac{\hbar^2 (\pi / L)^2}{2m} (n_x{}^2 + n_y{}^2 + n_z{}^2)$$

These are the allowed energy levels, or "quantum states"

Pauli exclusion principle: Each unique combination of n_x, n_y, n_z can only have two electrons (spin up, spin down).

Energy spectrum of free particles

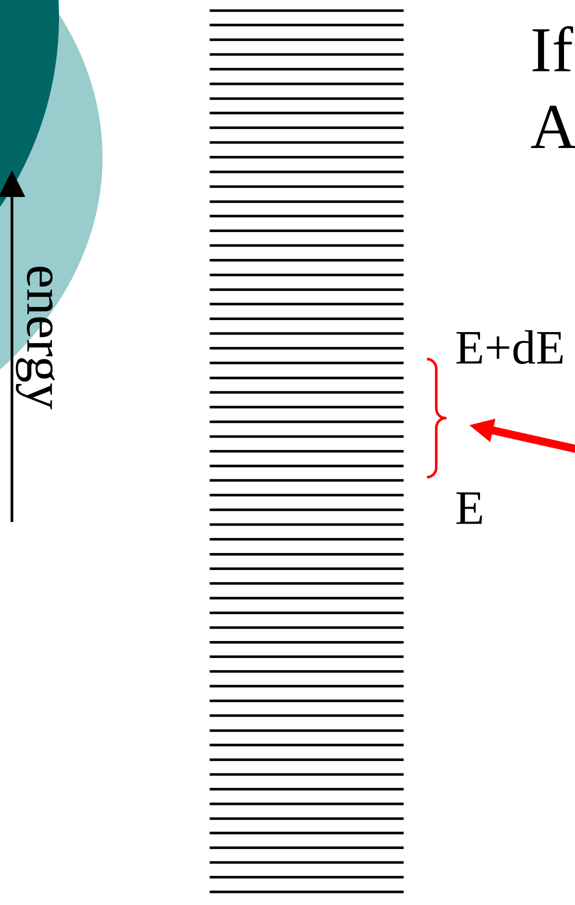


Etc.

$n_x=2, n_y=1, n_z=1$ | $n_x=1, n_y=2, n_z=1$ | $n_x=1, n_y=1, n_z=2$

$n_x=1, n_y=1, n_z=1$

Density of states



If L is large, states are very close together. Approximate as a continuum.

How many states?

$$N_E dE = ?$$

Number of states with energy between E and E + dE

$$\rho(E) dE = ?$$

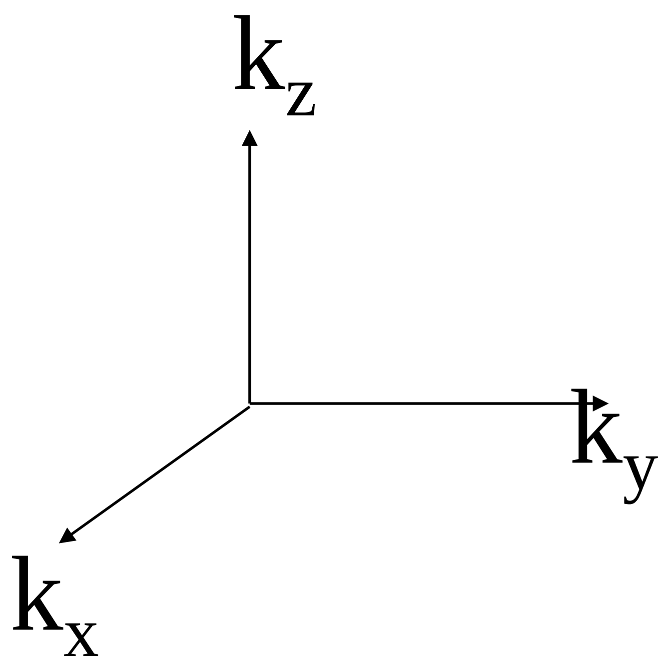
Number of states with energy between E and E + dE *per volume*.

Density of states

Easier first to think of in k-space:

Density of states in k-space is uniform:

One state per $(\pi/L)^3$:



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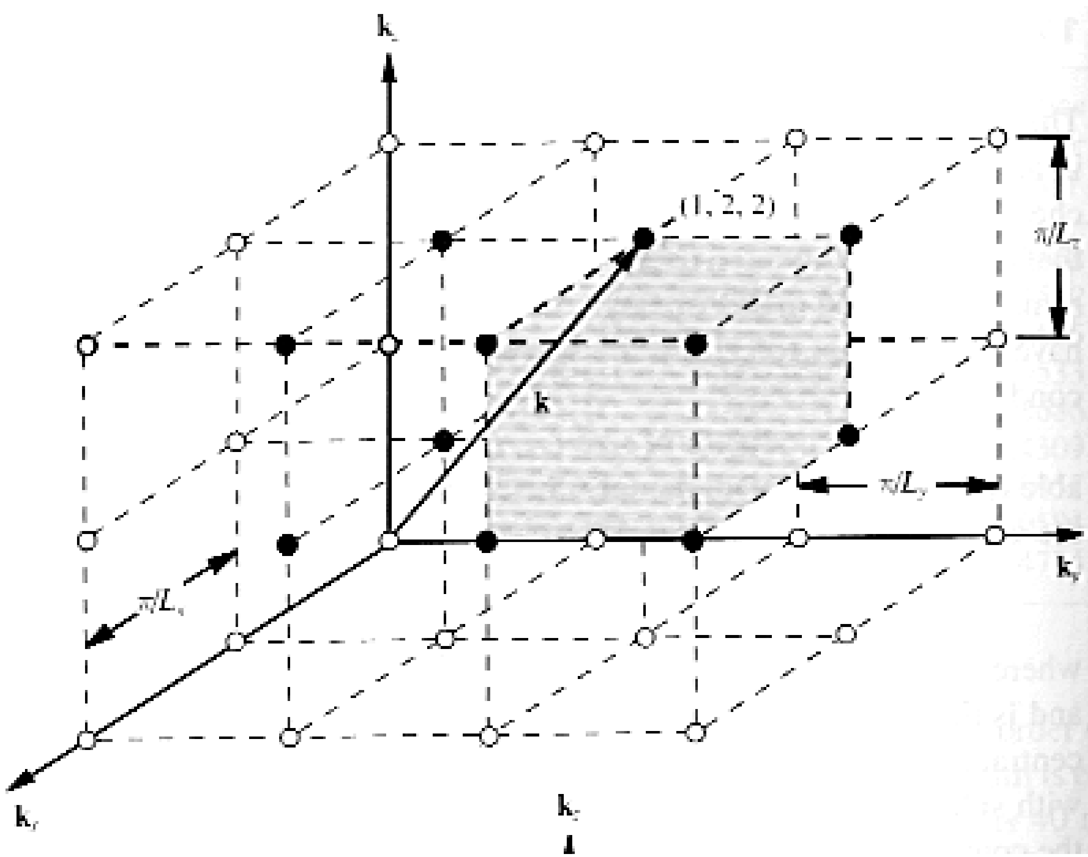
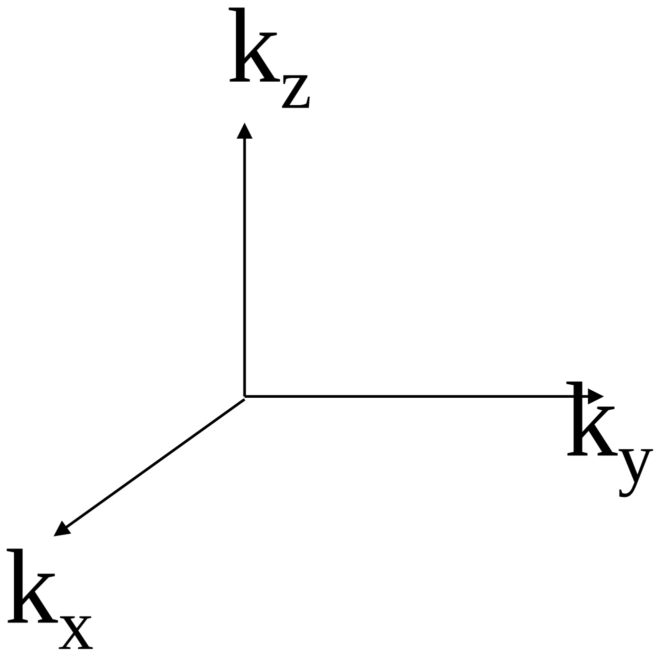


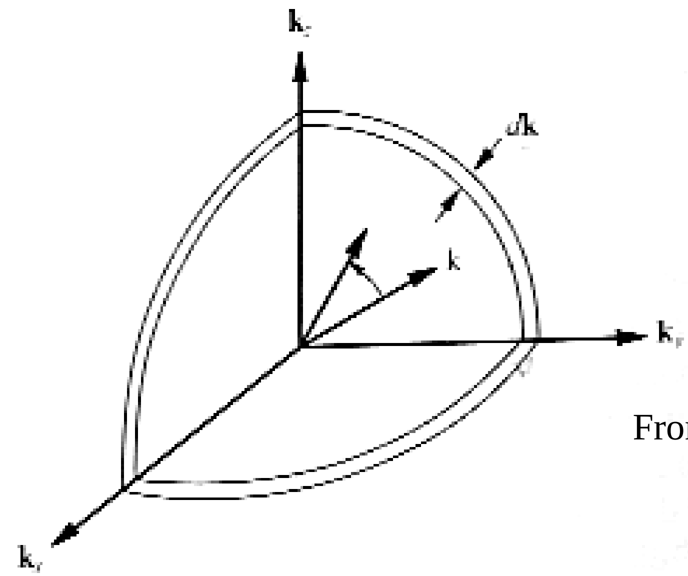
From Verdeyen

Density of states

Number of states between k, k+dk:

$$N_k dk = ?$$



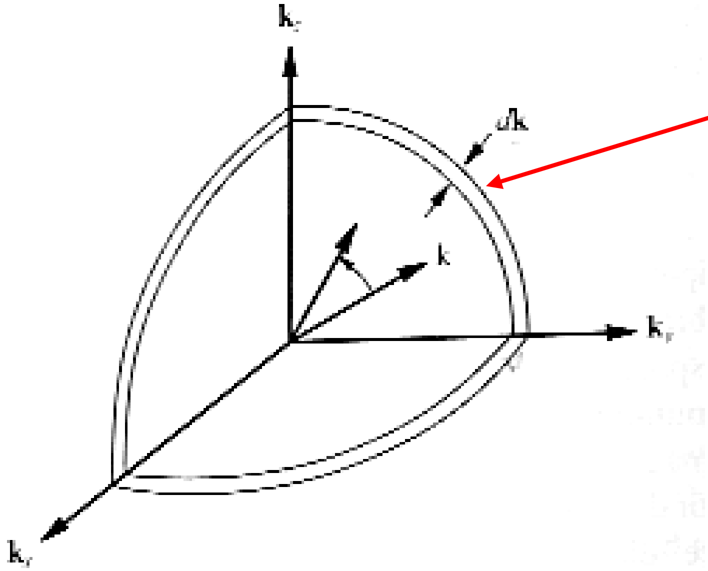


From Verdeyen

$$k_{n_x} = n_x \pi / L$$
$$k_{n_y} = n_y \pi / L$$
$$k_{n_z} = n_z \pi / L$$

$$k \equiv \sqrt{k_x^2 + k_y^2 + k_z^2}$$

$N_k dk = ?$



Volume of spherical shell
$=4\pi k^2 dk/8$
8 is for upper right quadrant

Number of states in volume=
Volume x States/volume

States/volume = $1 / (\pi/L)^3$:

$$N_k dk = \left(4\pi k^2 dk/8\right)\cdot\left(\frac{1}{(\pi/L)^3}\right)\cdot 2 = L^3 \frac{k^2 dk}{\pi^2}$$

$$\rho_k dk \equiv \frac{N_k dk}{\text{volume}} = \frac{k^2 dk}{\pi^2}$$

HW you will do calculation for 2 dimensional world.

$\rho(E)dE = ?$

We use:

$$\rho_k dk = \rho(E)dE$$

$$\rho_k dk = \frac{k^2 dk}{\pi^2}$$

$$E = \frac{\hbar^2 k^2}{2m} \Rightarrow k = \sqrt{\frac{2mE}{\hbar^2}} \Rightarrow dk = \sqrt{\frac{2m}{\hbar^2}} \frac{dE}{2\sqrt{E}}$$

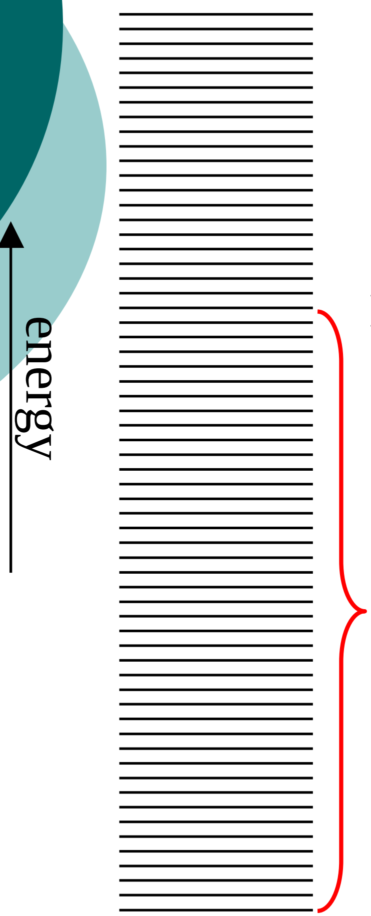
$$\rho(E)dE = \frac{2^{3/2} m^{3/2}}{\pi^2 \hbar^{3/2}} \cdot E^{1/2} dE$$

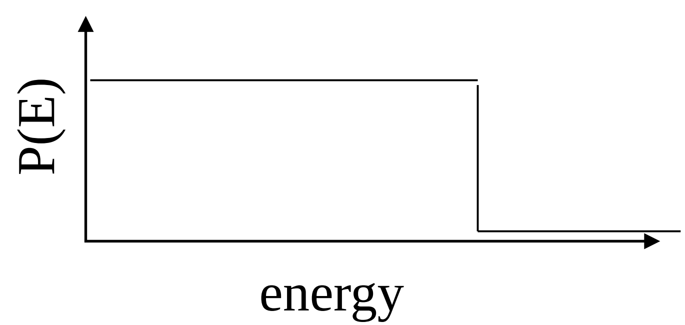
Fermi gas

At zero temperature, as we add electrons to the box, we gradually fill up all the states.
(DISCUSS PAULI EXCLUSION PRINCIPLE -IMPORTANT!)

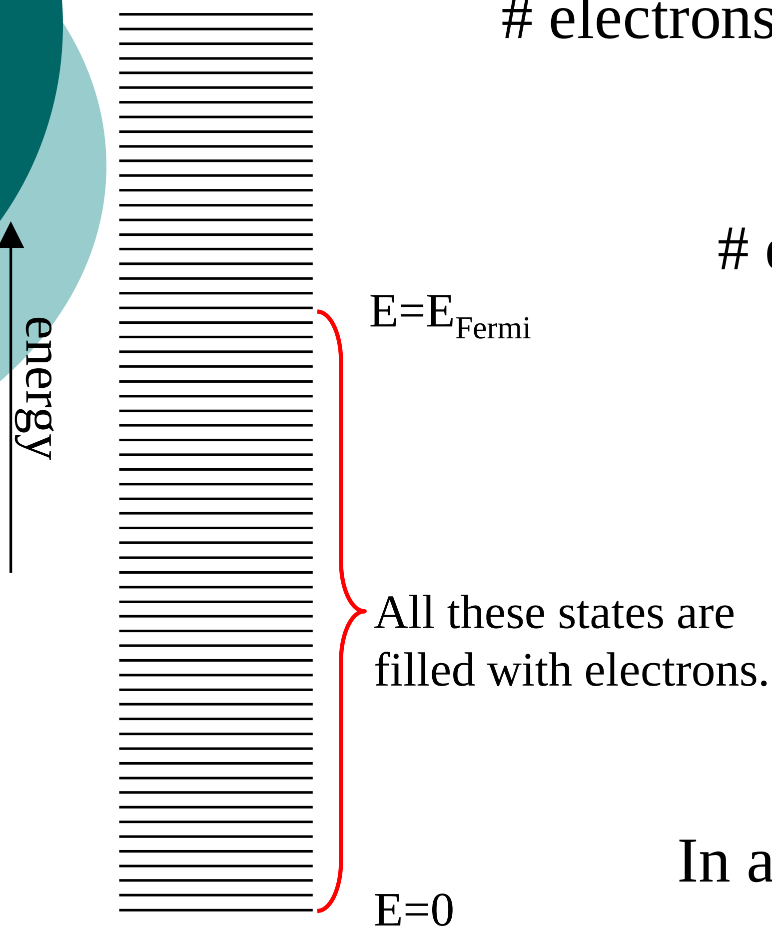
When we are done filling the box, the energy of the last electron is called the "Fermi energy."

"Gas" means we neglect electron-electron interactions.





Fermi energy



$$\#\ \text{electrons} = \int_0^{E_f} N_{\text{E}}\,\text{dE} = \int_0^{E_f} L^3 \frac{2^{1/2} m^{3/2}}{\pi^2 \hbar^{3/2}} \cdot E^{1/2} dE$$

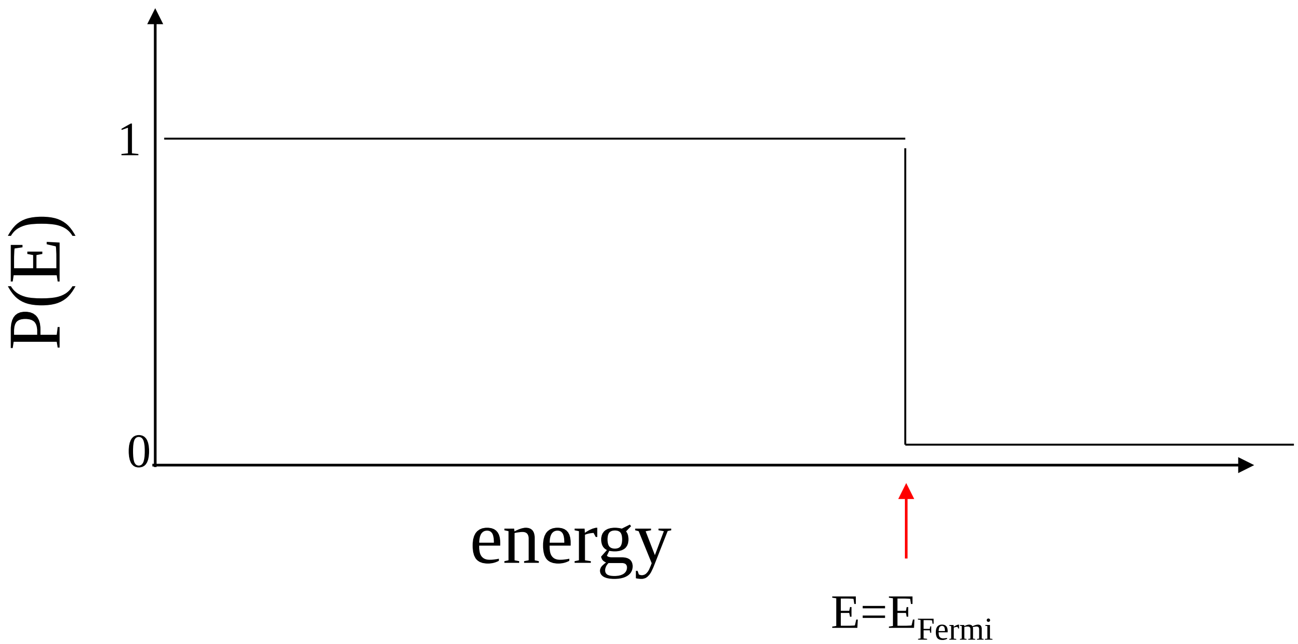
$$\#\ \text{electrons} = L^3 \frac{2^{1/2} m^{3/2}}{\pi^2 \hbar^{3/2}} \frac{2}{3} E_f^{3/2}$$

$$\Rightarrow E_f = \frac{\hbar^2 3^{2/3} \pi^{4/3}}{2m} \left(\frac{\#\ \text{electrons}}{L^3} \right)^{2/3}$$

In a typical metal, 1 electron /(0.1 nm)^3.

$E_f \sim 10$ eV

Occupation probability



P(E) = probability of occupying a state with energy E

What about finite temperature?

Boltzmann

Recall Boltzmann factor P(ε):

"The probability for a physical system to be in a state with energy ε is proportional to ."

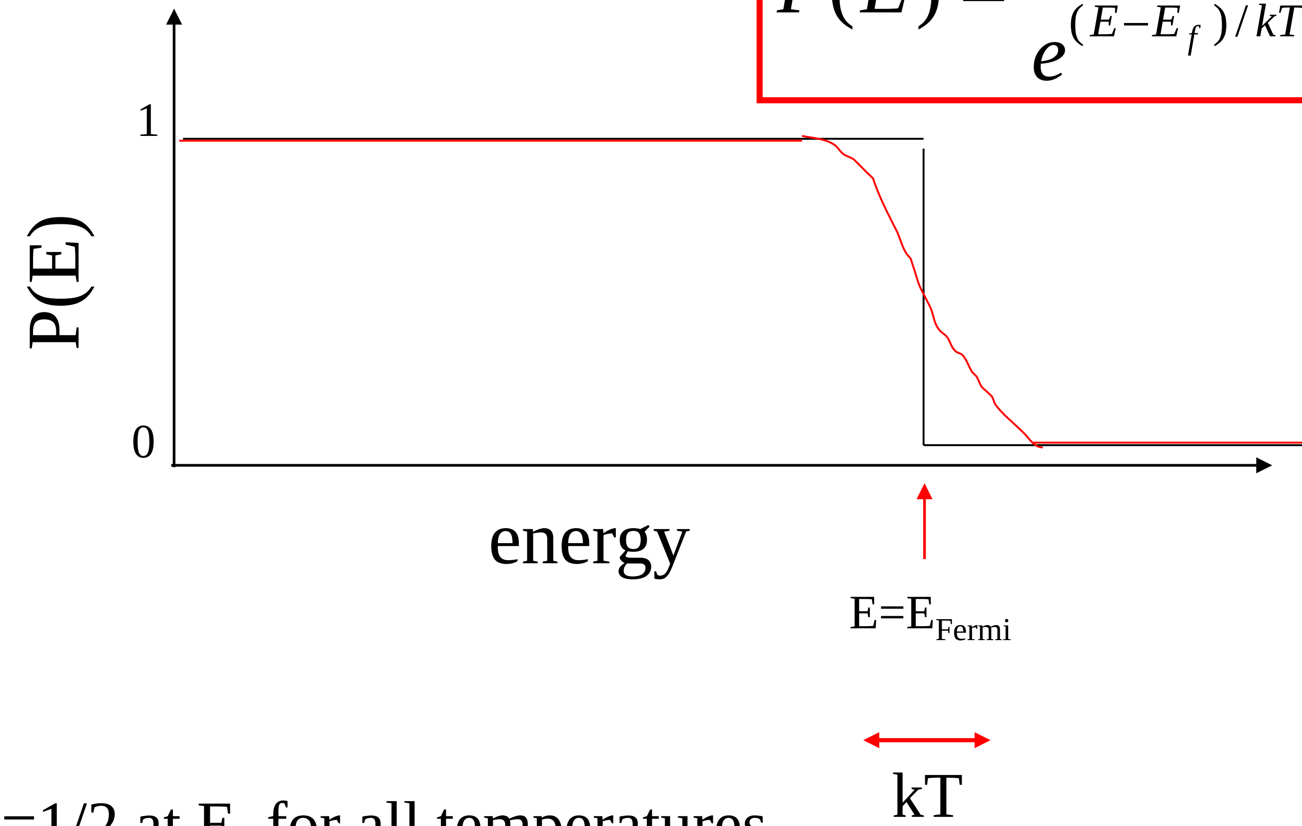
This is actually not quite true. It is classical.
A quantum calculation shows for electrons:

$$P(E) = \frac{1}{e^{(E-E_f)/kT} + 1}$$

Called Fermi-Dirac distribution function.
Boltzman is high-energy limit (discuss!)

Fermi-Dirac

$$P(E) = \frac{1}{e^{(E-E_f)/kT} + 1}$$



P=1/2 at E_f for all temperatures.